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Taylor’s expansion application in simplifying the groundwater analytical solution equations

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Abstract
The analytical modeling solutions and Taylor's expansion have been applied to various groundwater equations. This study explores the effectiveness of Taylor's expansion in simplifying the capture zone (CZ) equations for a multi-horizontal well (HW) system. The examined well system encompasses a combination of arbitrarily located production and recharge HWs, each with varying discharge or recharge rates and directions of uniform flow. To validate the utility of Taylor's expansion in this context, the velocity potential values from the equations were calculated both before and after applying the expansion. To further confirm the precision of the results, equipotential and flow lines were generated for both the original and simplified equations. The analysis revealed significant findings regarding accuracy: the percentage differences in hydraulic head between the original equations and those derived from Taylor's expansion reached peaks of 59.07% and 68.51% for aquifer thicknesses ranging from 0 to 50 meters. Remarkably, as the aquifer thickness increased, these percentage differences decreased substantially, achieving minimum values at a thickness of 1500 meters. Overall, the application of Taylor's expansion proves to be highly effective for aquifer thicknesses approximately 200 meters or greater in terms of hydraulic head and around 400 meters or more concerning the corresponding equations. The drawing of equipotential and flow lines reinforces the validity of these findings and demonstrates the practical applicability of the simplified equations. Furthermore, the established thresholds of thicknesses for accurate application provide critical guidance for future hydrological analyses. Future research could expand on these findings by exploring its applications under diverse geological conditions.

Keywords: Capture zone, Horizontal wells, Conceptual model, Mathematical equations.

Introduction

One of the ways to extract groundwater is to use horizontal wells (HWs), which provide a more useful surface for water exit due to having a long screen length compared to vertical wells (Sawyer & Lieuallen Dulam, 1998). The capture zone (CZ) of a HW is the area around the HW that supplies it with water (Barry et al., 2008). The equations related to the CZ around the HW for a multi-HW system have been previously obtained (Talebizadeh, 2024). In this paper, these equations can be simplified by using Taylor's expansion.

Taylor's series, named after a mathematician named Brooke Taylor, is a power series expansion of a function at a certain point. The derivatives evaluated at that point are used to construct this function, and each derivative adds a term to the series (Hammad, 2023). Taylor's series can estimate the value of a function at a nearby point, especially for evaluating functions

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whose derivatives are difficult to calculate (Schneider et al., 2018). For example, the sine and cosine functions can be expressed as Taylor's series, which allows them to be evaluated with arbitrary precision. Because of its convergence property, complex functions can be replaced with simpler polynomial functions. The advantage of this method is its high accuracy in approximating functions, which makes analysis and calculations easier. Another advantage of this method is its significant application in all areas of calculus, which can accurately convert nonlinear issues into linear problems (Wu et al., 2024). One practical application of this method is numerical simulations, in which Taylor expansion is used to determine the function's value at different points and reduce the complexity of the calculations. Another application of this method is to determine the limit, a function's extreme value, the value of the higher derivative at a certain point, the convergence of generalized integrals, an approximation, the proof of inequalities, and more (Weidun et al., 2005).

Many studies have been done on the use of Taylor's expansion. For example, a Taylor method for solving Fredholm integral equations was presented by Kanwall & Liu (1989), and then this method was generalized by Sezer (1994) for Volterra integral equations. Kesan (2003) has used Taylor's matrix method for the approximate solution of linear differential equations. Berna & Sezer (2010), used a Taylor polynomial approximation to solve hyperbolic partial differential equations with constant coefficients and gave several numerical examples to demonstrate the efficiency and reliability of the method. The solution of stochastic partial differential equations is also presented by Taylor's expansion (Jentzen, 2018). The propagation of uncertainty on a nonlinear measurement model is presented by Gu et al. (2021) using a higher-order Taylor's series. Taylor's expansion has also been used in groundwater problems. Marquardt (1963) introduced Taylor's expansion as one of the linearization methods of nonlinear equations in groundwater models. Michael et al. (1981) applied the first and second-order uncertainty analysis to the numerical groundwater flow models, which used Taylor's expansion for this issue and obtained the model equations. The results showed that these equations can estimate the mean and variance-covariance properties of piezometric head predictions, given corresponding statistics for aquifer parameters: material properties, initial conditions, boundary conditions, and inputs. Zhan (1999a) obtained an analytical solution of the Capture time for a particle moving from its initial location to a HW. Then, for comparison, he obtained the Capture time for a vertical well in an aquifer with the same characteristics as a HW, which he applied Taylor's expansion to his equation to simplify this problem. It should be noted that the aquifer thickness is considered infinite, which makes the effect of boundaries on the pumping well negligible, and the results showed that under such conditions, the performance of horizontal and vertical wells is similar. Grid refinement scheme in numerical groundwater flow models by increasing the accuracy of the solution without causing problems for run time of the model has been introduced by Mansour & Spink (2013), which is based on the theory of divergence and Taylor's expansion. The results showed that the use of more terms of Taylor's expansion improves the numerical solution and produces acceptable degrees of accuracy. Suk & Park (2019) have introduced a new numerical method for the accurate and efficient calculation of the Richards equation to simulate variably saturated flow in heterogeneous layered porous media. In the proposed method, the Kirchhoff integral transformation was applied. To avoid the dyadic characteristics at the material interface, a truncated Taylor series expansion was applied to the Kirchhoff head at the material interface. Accordingly, through the Taylor series expansion, a set of algebraic equations in the one-dimensional control volume finite difference discretized system formed a tridiagonal matrix system. The results clearly demonstrated that the approach was not only more computationally efficient but also more accurate and robust than other numerical methods.

The purpose of this research is to use Taylor's expansion to simplify equations for a multi-HWs system and to check its accuracy and sensitivity. The aquifer is considered confined and has a boundary on the left side that can be either in-flow or no-flow. Taylor's expansion makes the equations easy to use for any user. To demonstrate the validity of using Taylor's expansion the φ values for the equations obtained before and after using Taylor's expansion were calculated for three different positions of the HW, in four different positions of the plane, and for three different flow rates, and the results are compared. In the following, to confirm the correctness of the obtained equations, the equipotential and flow lines were drawn before and after simplification with Taylor's expansion and compared with each other.

Conceptual model

A schematic image of the confined aquifer with a boundary on the left side can be seen in Figure 1. This boundary can be in-flow (Figure 1(i)) or no-flow (Figure 1(ii)). The right side of the aquifer has an infinite boundary. Talebizadeh (2024) tried to solve this problem, using the theory of image wells and found the below equations (Equations 1, 2 and 3):

$$\zeta_D(u) = -Iu_D(\cos\alpha - i\sin\alpha) + \sum_{g=1}^{N} Q_{Dg}\left[\ln\left(\frac{\sin(G_{1D})}{\sin(G_{3D})}\right)\left(\frac{\sin(G_{2D})}{\sin(G_{4D})}\right)\right] \quad \text{(Equation 1)}$$

$G_{1D} = \pi(u_D - a_{Dg} - ib_{Dg})/2$, $G_{2D} = \pi(u_D + a_{Dg} - ib_{Dg})/2$, $G_{3D} = \pi(u_D - a_{Dg} + ib_{Dg})/2$
$G_{4D} = \pi(u_D + a_{Dg} + ib_{Dg})/2$

$$\varphi_D = -I(x_D\cos\alpha + z_D\sin\alpha) + \sum_{g=1}^{N} \frac{Q_{Dg}}{2}\left[\ln\left[\frac{(M_{1D} - M_{2D})(M_{1D} - M_{3D})}{(M_{4D} - M_{2D})(M_{4D} - M_{3D})}\right]\right] \quad \text{(Equation 2)}$$

$M_{1D} = \cosh(\pi(z_D - b_{Dg}))$, $M_{2D} = \cos(\pi(x_D - a_{Dg}))$, $M_{3D} = \cos(\pi(x_D + a_{Dg}))$,
$M_{4D} = \cosh(\pi(z_D + b_{Dg}))$

$$\psi_D = -I(z_D\cos\alpha - x_D\sin\alpha) + \sum_{g=1}^{N} Q_{Dg}c_g\left[\tan^{-1}\left[\frac{P_{1D}}{P_{2D}}\right] + \tan^{-1}\left[\frac{P_{1D}}{P_{3D}}\right] - \tan^{-1}\left[\frac{P_{4D}}{P_{2D}}\right] - \tan^{-1}\left[\frac{P_{4D}}{P_{3D}}\right]\right] \quad \text{(Equation 3)}$$

$P_{1D} = \tanh(\pi((z_D - b_{Dg})/2)$, $P_{2D} = \tan(\pi((x_D - a_{Dg})/2)$, $P_{3D} = \tan(\pi((x_D + b_{Dg})/2)$,
$P_{4D} = \tanh(\pi((z_D + b_{Dg})/2)$

And boundary configuration in Figure 1(ii) as follows (Equations 4, 5, 6 and 7):

$$\zeta_D(u) = -Iu_D(\cos\alpha - i\sin\alpha) + \sum_{g=1}^{N} Q_{Dg}\left[\ln\left[\sin(G_{1D})\sin(G_{2D})\sin(G_{3D})\sin(G_{4D})\right]\right] \quad \text{(Equation 4)}$$

$$\varphi_D = -I(x_D\cos\alpha + z_D\sin\alpha) + \sum_{g=1}^{N} \frac{Q_{Dg}}{2}\left[\ln\left[(M_{1D} - M_{2D})(M_{1D} - M_{3D})(M_{4D} - M_{2D})(M_{4D} - M_{3D})\right]\right] \quad \text{(Equation 5)}$$

$$\psi_D = -I(z_D\cos\alpha - x_D\sin\alpha) + \sum_{g=1}^{N} Q_{Dg}\left[\tan^{-1}\left[\frac{P_{1D}}{P_{2D}}\right] + \tan^{-1}\left[\frac{P_{1D}}{P_{3D}}\right] + \tan^{-1}\left[\frac{P_{4D}}{P_{2D}}\right] + \tan^{-1}\left[\frac{P_{4D}}{P_{3D}}\right]\right] \quad \text{(Equation 6)}$$

Taylor's series is a power series expansion of a function at a certain point. The Taylor's series around a particular point is given as follows (Hammad, 2023):

$$f(s)=\sum_{n=0}^{\infty}\frac{f^{n}(c)}{n!}(s-c)^{n}$$ (Equation 7)

where f^{n} is the nth derivative of *f* evaluated at the point *c*, *c* is the real or complex number, *n* is the summation number and *s* is a variable.

Governing mathematical equations

Taylor's series is a power series expansion of a function in a certain point that is used in various groundwater problems. Talebizadeh, 2024 has derived the analytical solution of the CZ equations for boundary configuration shown in Figures 1(i) and 1(ii). These solutions are long and may be difficult to use for any users. In this research, we use Taylor's expansion to simplify them so that will be applicable and understandable for any users without complicated mathematics (Zhan, 1999a). For this purpose, the following Taylor’s expansion is considered (Equation 8) (Taylor, 1976):

$$\sin(x)\approx x \qquad \cos(x)\approx 1-\frac{x^{2}}{2} \qquad \cosh(x)\approx 1+\frac{x^{2}}{2} \qquad \tan(x)\approx x \qquad \tanh(x)\approx x$$ (Equation 8)

If we considered *x* in Equation 1 as G_{1D}, G_{2D}, G_{3D}, and G_{4D}, according to Equation 8, can be replaced as sin (G_{1D}), sin (G_{2D}), sin (G_{3D}), and sin (G_{4D}) in Equation 1, therefore (Equation 9):

$$\zeta_{D}(u)=-Iu_{D}(\cos\alpha-i\sin\alpha)+\sum_{g=1}^{N}Q_{Dg}\left[\ln\left(\frac{(G_{1D})}{(G_{3D})}\right)\left(\frac{(G_{2D})}{(G_{4D})}\right)\right]$$ (Equation 9)

To simplify Equation 2 by using Taylor's expansion, first, the *ln* in the equation is expanded and the values M_{1D} to M_{4D} are replaced. Which results in (Equation 10):

$$\varphi_{D}=-I(x_{D}\cos\alpha+z_{D}\sin\alpha)+\sum_{g=1}^{N}\frac{Q_{Dg}}{2}\Big[\big[\ln\cosh(\pi(z_{D}-b_{Dg}))-\cos(\pi(x_{D}-a_{Dg}))\big]$$
$$+\big[\ln\cosh(\pi(z_{D}-b_{Dg}))-\cos(\pi(x_{D}+a_{Dg}))\big]-\big[\ln\cosh(\pi(z_{D}+b_{Dg}))-\cos(\pi(x_{D}-a_{Dg}))\big]-$$
$$\big[\ln\cosh(\pi(z_{D}+b_{Dg}))-\cos(\pi(x_{D}+a_{Dg}))\big]\Big]$$
(Equation 10)

If $\pi(z_{D}-b_{Dg})$, $\pi(x_{D}-a_{Dg})$, $\pi(x_{D}+a_{Dg})$ and $\pi(z_{D}+b_{Dg})$ are considered as *x* in this Equation (10) and according to Equation 8, substituting $\cosh(x)$ and $\cos(x)$ gives the following Equation (11):

$$\varphi_{D}=-I(x_{D}\cos\alpha+z_{D}\sin\alpha)+\sum_{g=1}^{N}\frac{Q_{Dg}}{2}\left[\left[\ln(1+\frac{(\pi(z_{D}-b_{Dg}))^{2}}{2}-(1-\frac{(\pi(x_{D}-a_{Dg}))^{2}}{2}))\right]\right.$$
$$+\left[\ln(1+\frac{(\pi(z_{D}-b_{Dg}))^{2}}{2}-(1-\frac{(\pi(x_{D}+a_{Dg}))^{2}}{2}))\right]-\left[\ln(1+\frac{(\pi(z_{D}+b_{Dg}))^{2}}{2}-(1-\frac{(\pi(x_{D}-a_{Dg}))^{2}}{2}))\right]-$$
$$\left.\left[\ln(1+\frac{(\pi(z_{D}+b_{Dg}))^{2}}{2}-(1-\frac{(\pi(x_{D}+a_{Dg}))^{2}}{2}))\right]\right]$$
(Equation 11)

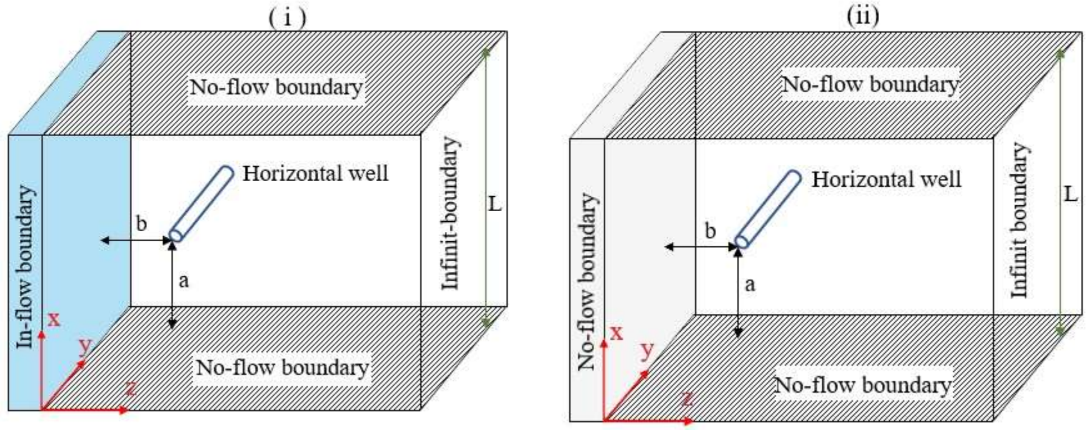

Figure 1. Schematic cross-section of confined aquifer boundary configurations. (i) bounded in left by constant head inflow boundary, (ii) bounded in left by no-flow boundary. The right boundary has an infinite extent. The well is horizontal at location (a, b). The aquifer thickness is L, and the origin of the x-z axis is on the lower left part of the aquifer

In Equation 11, the term inside *ln* is written in a simpler form by taking the common denominator, and finally, *ln* can also be written as follows (Equation 12):

$$\varphi_D = -I\left(x_D \cos\alpha + z_D \sin\alpha\right) + \sum_{g=1}^{N} \frac{Q_{Dg}}{2}\left[\ln\left(\frac{(F_{1D})(F_{2D})}{(F_{3D})(F_{4D})}\right)\right] \qquad \text{(Equation 12)}$$

$$F_{1D} = \pi^2(z_D - b_{Dg})^2 + \pi^2(x_D - a_{Dg})^2/2, \qquad F_{2D} = \pi^2(z_D - b_{Dg})^2 + \pi^2(x_D + a_{Dg})^2/2$$

$$F_{3D} = \pi^2(z_D + b_{Dg})^2 + \pi^2(x_D - a_{Dg})^2/2 \qquad F_{4D} = \pi^2(z_D + b_{Dg})^2 + \pi^2(x_D + a_{Dg})^2/2$$

To simplify Equation 3, based on Equation 8, $\tanh(x)$ and $\tan(x)$ are considered equal to x, where x is $\pi((z_D - b_{Dg})/2$, $\pi((x_D - a_{Dg})/2$, $\pi((x_D + b_{Dg})/2$ and $\pi((z_D + b_{Dg})/2$ in Equation 3. According to these cases, Equation 3 can be expressed as follows (Equation 13):

$$\psi_D = -I(z_D \cos\alpha - x_D \sin\alpha) + \sum_{g=1}^{N} Q_{Dg}\left[tan^{-1}\left(\frac{\frac{\pi((z_D - b_{Dg})}{2)}}{\frac{\pi((x_D - a_{Dg})}{2)}}\right) + tan^{-1}\left(\frac{\frac{\pi((z_D - b_{Dg})}{2)}}{\frac{\pi((x_D + a_{Dg})}{2)}}\right) - tan^{-1}\left(\frac{\frac{\pi((z_D + b_{Dg})}{2)}}{\frac{\pi((x_D - a_{Dg})}{2)}}\right) - tan^{-1}\left(\frac{\frac{\pi((z_D + b_{Dg})}{2)}}{\frac{\pi((x_D + a_{Dg})}{2)}}\right)\right] \qquad \text{(Equation 13)}$$

This equation can be simplified as follows (Equation 14):

$$\psi_D = -I(z_D \cos\alpha - x_D \sin\alpha) + \sum_{g=1}^{N} Q_{Dg}\left[\tan^{-1}\left(\frac{E_{1D}}{E_{2D}}\right) + \tan^{-1}\left(\frac{E_{1D}}{E_{3D}}\right) - \tan^{-1}\left(\frac{E_{4D}}{E_{2D}}\right) - \tan^{-1}\left(\frac{E_{4D}}{E_{3D}}\right)\right]$$

(14)

$$E_{1D} = \pi(z_D - b_{Dg}), \qquad E_{2D} = \pi(x_D - a_{Dg}), \qquad E_{3D} = \pi(x_D + a_{Dg}), \qquad E_{4D} = \pi(z_D + b_{Dg})$$

Similarly, Equations 4 to 6, which are related to the boundary configuration in Figure 1(ii), can be simplified using Taylor's expansion respectively as (Equations 15, 16 and 17):

$$\zeta_D(u) = -Iu_D(\cos\alpha - i\sin\alpha) + \sum_{g=1}^{N} Q_{Dg}\left[\ln((G_{1D})(G_{2D})(G_{3D})(G_{4D}))\right] \quad \text{(Equations 15)}$$

$$\varphi_D(u) = -I\left(x_D\cos\alpha + z_D\sin\alpha\right) + \sum_{g=1}^{N} \frac{Q_{Dg}}{2}\left[\ln\left((F_{1D})(F_{2D})(F_{3D})(F_{4D})\right)\right] \quad \text{(Equations 16)}$$

$$\psi_D = -I(z_D\cos\alpha - x_D\sin\alpha) + \sum_{g=1}^{N} Q_{Dg}\left[\tan^{-1}\left(\frac{E_{1D}}{E_{2D}}\right) + \tan^{-1}\left(\frac{E_{1D}}{E_{3D}}\right) + \tan^{-1}\left(\frac{E_{4D}}{E_{2D}}\right) + \tan^{-1}\left(\frac{E_{4D}}{E_{3D}}\right)\right]$$

(Equations 17)

Therefore, Equations 15 to 17 are the simplified form of Equations 4 to 6 using Taylor's expansion. It should be noted that due to the similarity of the calculation process, details have been avoided.

Results and discussion

In this section, the accuracy of Taylor's expansion is investigated. Due to the large number of points, only several HWs with several different flow rates at different positions in the plane are considered and the results are presented in the form of a graph.

Taylor's expansion accuracy

Simplifying equations using Taylor's expansion may introduce errors in some conditions. In the following, the difference percentage of the calculated equations (Equations 1 to 6) (Talebizadeh, 2024) and the simplified equation with Taylor's expansion (Equations 9, 12, 14, 15, 16, and 17) have been investigated; For example, Equation 2 is compared with its equivalent obtained from Taylor's expansion; i.e. Equation 12. For this purpose, three HW's in different locations, i.e. (*a* = 20*, b* = 60), (*a* = 70*, b* = 45) and (*a* = 50*, b* = 30) are considered. For each HW, the values of φ at four positions on the plane (*x, z* = 20, *x, z* = 50, *x, z* = 70, and *x, z* = 100) and three different flow rates (*Q* = 345.6 *m/day, Q* = 502.6 *m/day* and *Q* = 864 *m/day*) were calculated before and after using the Taylor expansion (Equations 2 and 3, respectively). The thickness used for the aquifer in these calculations ranges from *L = 30m* to *L = 2300m.* It should be noted that these calculations were performed for both $\alpha = 90^\circ$ and $\alpha = 270^\circ$. After calculating φ, hydraulic head values (*h*) were obtained with the equation of $\varphi = Kh$ and the head became dimensionless by using $h_D = \frac{h}{L}$ (Bear, 1972). Then, the difference percentage of h_D calculated using Equations 2 and 12 was obtained and the graph of aquifer thickness *(L)* changes against this difference percentage was drawn separately for $\alpha = 90^\circ$ and $\alpha = 270^\circ$ (Figures 2 and 3). Table 1 shows the brief of Figures 2 and 3 data (the average difference percentage calculated from Equations 2 and 12 for different thicknesses). As seen in Figures 2 and 3 and in Table 1, for both values of α, the average percentage difference h_D in Equations 2 and 12 is highest in thicknesses from 0 to 50 m, which is %59.07 and %68.51 for $\alpha = 90^\circ$ and $\alpha = 270^\circ$, respectively. These results show that when the aquifer thickness is less than 50m, Taylor's expansion is not very accurate. As the thickness increases (Table 1), this difference percentage decreases significantly, reaching its lowest value after the thickness of 1500 m and reaches at its lowest values, i.e., %0.008 for $\alpha = 90^{\circ}$ and %0.009 for $\alpha = 270^{\circ}$. Based on the Table 1, from a thickness of about 200 m for $\alpha = 90^{\circ}$ and 400 m for $\alpha = 270^{\circ}$, the simplified Taylor's expansion equation can be used with high accuracy instead of the equation calculated by Talebizadeh (2024). These figures also demonstrate that users with a specified aquifer thickness could observe the different percentage of h_D and the validity of Taylor's expansion.

Capture zone of one arbitrarily located HW near one boundary

In this section, the equations of φ_D and ψ_D representing the net flow in the x-z plane, in two cases before and after simplification with Taylor's expansion, for one HW (Figures 4(A) and (B)) and three HWs (Figures 5(A) and (B)) are drawn and compared against each other for the boundary configuration shown in Figure 1(i).

Table 1. Difference percentage h_D Eq.2 and h_D Eq. 12 for α = 90º and 270°

L (m)	% Difference h_D Eq.2 and h_D Eq. 12α = 90°	% Difference h_D Eq.2 and h_D Eq. 12 α = 270°
0 - 50	59.06	68.51
50 - 100	35.70	58.26
100 - 200	9.52	37.75
200 - 300	2.85	13.20
300 - 700	0.38	0.88
700 - 1500	0.05	0.08
1500 - 2300	0.008	0.009

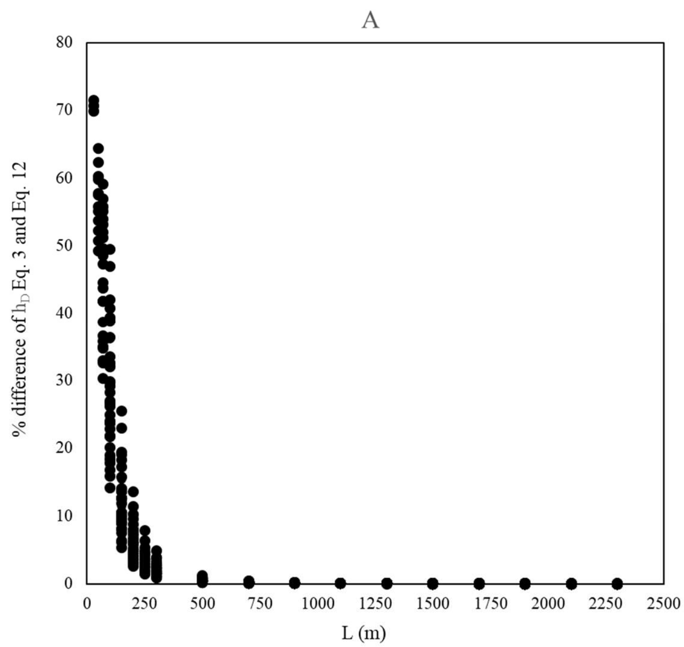

Figure 2. The graph of the difference percentage of h_D calculated by using Eq. 2 and Eq. 12 against the changes in aquifer thickness (L) for $\alpha = 90^{\circ}$, three different positions of the HW, i.e. (*a* = 20*, b* = 60), (*a* = 70*, b* = 45) and (*a* = 50*, b* = 30), in four different location of the plane (*x, z* = 20, *x, z* = 50, *x, z* = 70 and *x, z* = 100) and for three different HW flow rates (*Q* = 345.6 *m/day, Q* = 502.6 *m/day* and *Q* = 864 *m/day*)

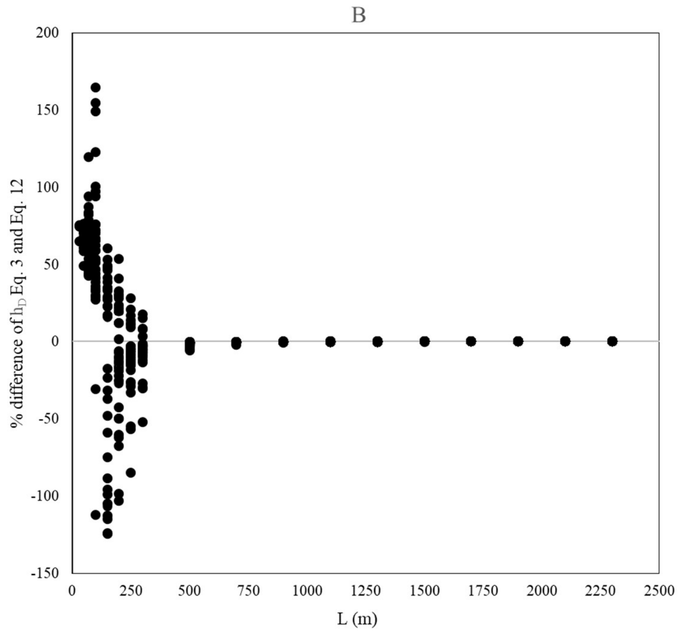

Figure 3. The graph of the percentage difference of h_D calculated by using Eq. 3 and Eq. 12 against the changes of L for $\alpha = 270^{\circ}$, three different positions of the HW, i.e. (a = 20, b = 60), (a = 70, b = 45) and (a = 50, b = 30), in four different location of the plane (x, z = 20, x, z = 50, x, z = 70 and x, z = 100) and for three different HW flow rates (Q = 345.6 m/day, Q = 502.6 m/day and Q = 864 m/day)

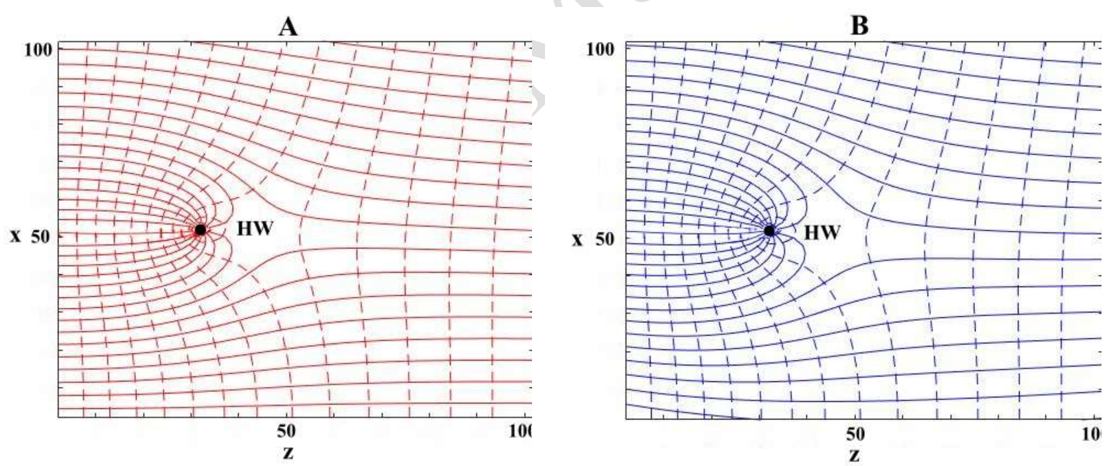

Figure 4. Velocity potential (dashed line) and stream function (solid line) of a HW: (A) before simplifying the equations with Taylor's expansion, (B) after simplifying the equations with Taylor's expansion. The solid circle is the extraction HW

Parameters φ_D and ψ_D, show equipotential and streamlines, respectively. The flow lines converged toward the HW, which indicates the CZ (Zarei-Dodeji & Samani, 2016). Equations 12 and 13 were used to draw Figures 4 and 5(A) and Figures 4 and 5(B) have been drawn using Equations 23 and 26. In these Figures, the thickness of the aquifer is considered to be 250 m and the direction of uniform flow is from left to the right $(\alpha = 90^{\circ})$. In the Figures 4(A) and (B) the position of the HW is at $a = 50(m)$, $b = 30(m)$, and its extraction rate is *4 lit/s*. In Figures 5 (A) and (B) the position of HW is $a_1 = 30(m), b_1 = 65(m), a_2 = 60(m), b_2 = 65(m)$ and $a_3 = 40(m), b_3 = 20(m)$ and $Q_1 = Q_2 = 5lit/s$ and $Q_3 = -5lit/s$.

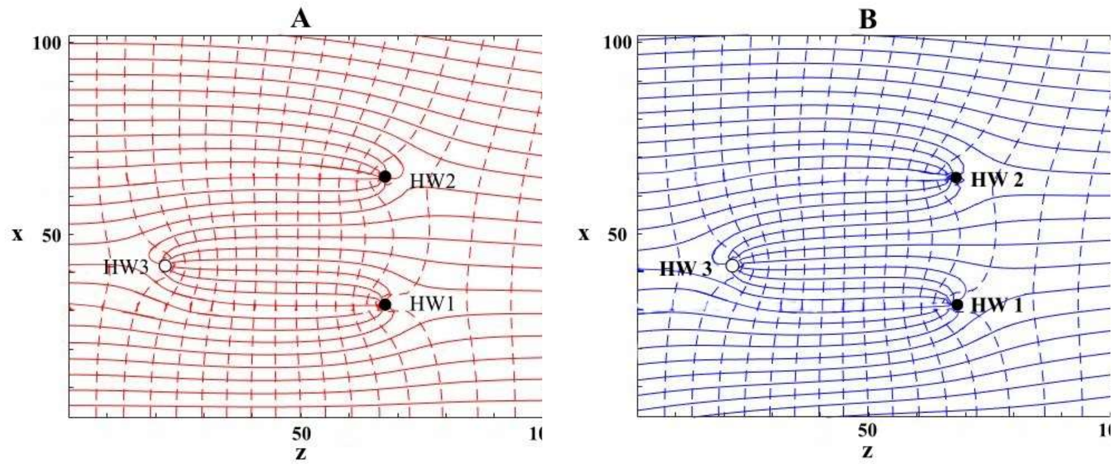

Figure 5. Velocity potential (dashed line) and stream function (solid line) of three HWs: (A) before simplifying the equations with Taylor's expansion, (B) after simplifying the equations with Taylor's expansion. The solid circle is the extraction HW and the hollow circle is the injection HW

As illustrated in these figures, it is clear that the HW takes its water from the left inflow boundary. As can be seen, the streamlines are parallel to the no-flow boundaries and come toward the HW, which is expected in this context. Also, it can be seen that Figures 4(A) and (B) and Figures 5(A) and (B) are similar. Based on this, from the thickness of about 200 m, the simplified Taylor's expansion equation can be used with high accuracy instead of the equation calculated in this research.

Conclusion

In this research, Taylor's expansion is used as a method to simplify CZ equations for a multi-HWs system obtained by Talebizadeh (2024) and the results were compared. The results show that for both values of α, the average difference percentage of h_D is the highest in thicknesses from 0 to 50 m, which is %59.07 and %68.51 for $\alpha = 90^\circ$ and $\alpha = 270^\circ$, respectively. As the thickness increases, this difference percentage decreases significantly, so that after the thickness of about 1500 m, it reaches its lowest value, i.e., %0.008 for $\alpha = 90^\circ$ and %0.009 for $\alpha = 270^\circ$. Based on this, from a thickness of about 200 m for $\alpha = 90^\circ$ and from a thickness of about 400 m for $\alpha = 270^\circ$ the simplified Taylor's expansion equation can be used with high accuracy instead of the equation calculated by Talebizadeh (2024). Drawing the equations also confirmed this issue. This research highlights the effective application of Taylor's expansion for simplifying the equations that describe the capture zone (CZ) of horizontal wells (HWs) in multi-HW systems. The simplification process results in user-friendly equations that facilitate practical applications in groundwater management without requiring extensive mathematical expertise. The accuracy assessment indicates that the Taylor expansion provides reliable approximations for aquifer thicknesses of around 200m and greater, with minimal percentage differences compared to traditional equations. However, it is crucial to note that for thicknesses under 50 m, the approach introduces significant errors, emphasizing the need for caution in its application under such conditions. Through graphical analysis of equipotential lines and streamlines, the study confirms that the simplified equations maintain the integrity of flow dynamics and capture zone characteristics. This reinforces the potential of Taylor's expansion as a valuable tool for hydrologists and water resource managers, enabling more efficient modeling of groundwater behaviors. The findings advocate for the integration of Taylor's expansion into groundwater flow modeling practices, enhancing both computational efficiency and accuracy. Future investigations should aim to evaluate the method's applicability across varying geological and hydrological contexts, thereby broadening its utility in diverse groundwater management scenarios.

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